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Preface to the first edition.

The present edition of Geometry is intended for the use of students taking up Mathematics in the Intermediate Examination of the Punjab University. The books generally in use in the Colleges here cover much more ground than that covered by the Syllabus laid down by the Mathematical Board and are on that account often perplexing to the average student from the prolixity of the matter dealt with in them. In preparing the present volume, therefore, I have kept in view the Syllabus prescribed by the University, and have endeavoured to follow the lines suggested as closely as possible. The present book never aims at supplanting the standard works on the subject, but is only a *vade mecum* to be used by the student for the purposes of his examination. In order to increase its value I have taken care to insert in the book a number of Questions selected from examination papers, and where necessary have given hints for their solution. Any suggetions to make the book more useful will be welcome; and would be gladly incorporated in future editions should such editions be ever in demand. Typographical errors which, inspite of careful proof-reading, may have crept in will also, if pointed out to the author or the publishers be thankfully corrected. In the end, I shall consider my labours well-spent if those for whom the book is meant derive any benefit from it.

My thanks are due to Lala Brij Lal Puri, B. A., for the kind readiness with which he read through the proofs, while the book was in the Press; as, without his timely assistance there would have been some delay in its publication.

D. A. V. College,
Lahore
Dated 21st September, 1912.

K. M. GHOSH.

Preface to the third edition.

It this edition the book has been thoroughly revised and the serial number of the propositions has been changed.

I am greatly indebted to S. N. Das Gupta Esq., M. A., Professor of Mathematics, F. C. College Lahore, who kindly revised the book and made some very valuable changes and additions.

D. A. V. College,
Lahore.
September. 1914.

K. M. Ghosh.

ERRATA.

Page 7 Step 2. **1st** line *for* $\angle \alpha R$ ***read*** $\angle R$

,, 8 Prop, 6 Figure—the line SO appears to be broken between μ and O

Page 24 *for* **Prop** 6 *read* **Prop.** 16

,, 27 After the exercises and before Prop. 17 add the following :—

Homothetic Figures—Two figures are said to be *homothetic* when the straight line joining every two points of the one is parallel to the straight line joining the corresponding points of the other.

"Two such figures are sometimes said to be similiar and similarly placed"—*Lachlan.*

[*Derivation*—From *homo* = same and *thesis* = to place to set.]

Page 28 line 11 *for* "$\therefore$" *read* "$\because$"

PUNJAB UNIVERSITY SYLLABUS FOR INTERMEDIATE EXAMINATION PLANE GEOMETRY OF THE STRAIGHT LINE AND CIRCLE.

As for the Matriculation Examination with the following additions:—

If A, B, C, D, be four points taken in order on a st. line

$AB \,.\, CD + BC \,.\, AD + CA \,.\, BD = 0$. (Prop. 22)

In any triangle, the perpendiculars at the middle points of the sides (Prop. 1); the perpendiculars from the vertices on the opposite sides (Prop. 4), the bisectors of the angles (Prop. 2), and the medians, are severally concurrent. (Prop. 3)

The feet of the perpendiculars on the sides of a triangle from any point on the circumcircle are collinear (Prop. 8).

The existence and simpler properties of the nine points circle. (Props. 5 & 6)

If ABC be a triangle and AD a median

$AB^2 + AC^2 = 2BD^2 + 2AD^2$ (Prop. 7)

If ABC be a triangle and AD the bisector of an angle

$BA \,.\, AC = BD \,.\, DC + AD^2$. (Prop. 9)

If ABC be a triangle and AD the perpendicular on the base from A, and AE the diameter of the circumcircle through A.

$BA \,.\, AC = EA \,.\, AD$. (Prop. 10).

ABCD be a quadrilateral inscribed in a circle

$AC \,.\, BD = AB \,.\, CD + BC \,.\, AD$ (Prop. 11) .

Determination of radical axis (Prop. 20) and radical centre of circles (Prop. 21) the locus of a point, the ratio of whose distances from two given points is equal to a given ratio is a circle. (Prop. 16)

If two rectilinear figures be similar they can be placed, so that the lines joining their corresponding vertices are concurrent. (Prop. 17)

If two rectilinear figures be similar, their corresponding sides and diagonals are proportional. (Prop. 18)

Arcs of a circle are proportional to the angles subtended by them at the centre. (Prop 19)

If three concurrent st. lines be drawr from the angular points of a triangle to meet the opposite sides, the product of the three alternate segments taken in order is equal to the product of the other three segments (Ceva's Theorem), and conversely. (Prop. 12 and 13.)

If a transversal cut the three sides of a triangle the product of the three alternate segments taken in order is equal to the product of the other three segments (Menelaus' Theorem) and conversely. (Prop. 14 and 15)

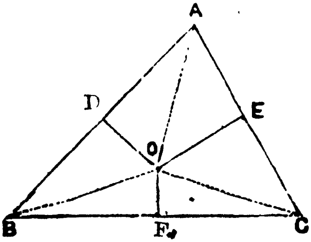
PLANE GEOMETRY

Concurrence of Certain lines.

Proposition 1.

In any triangle, the perpendiculars drawn at the middle points of the sides are concurrent.

Let ABC be a triangle and D,E,F the mid-points of AB, AC and BC. Then the ⊥s to the sides at D,E,F shall be concurrent.



From D and E draw perps. to AB and AC and let them meet at O. Join OF.

It is required to prove that OF *is* ⊥ *to* BC.

Joint OA, OB, OC.

Proof. **1.** Then ∵ O lies on the right bisector of AB, ∴ OA=OB.

Similarly OA=OC. ∴ OB=OC.

2. Now in the △s OBF, OCF, ∵ BF=CF (Hyp.); OF is common and OB=OC (proved);

∴ ∠OFB=∠ OFC = one Rt. angle.

3. ∴ OF is ⊥ BC.

Hence the three ⊥s OD, OE, OF meet at the same point O.

Note.—O is the **circum-centre** of △ ABC.

Exercises.

1. Through A,B,C draw parallels to the opposite sides forming the triangle A′ B′ C′. Prove, from this triangle, that the altitudes of the triangle ABC are concurrent.

2. Two triangles have one side of the one equal to one side of the other, and have the angles opposite to these sides supplementary; prove that they have equal circum-radii.

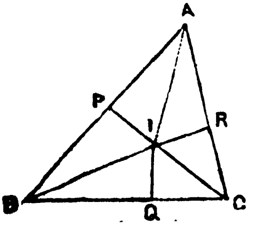
3. Prove that the circumradii of similar triangles are in ratio of their corresponding sides and hence shew that similar triangles are to one another as the squares of their circumradii.

4. The base and vertical angle of a triangle being given, the circumradius is constant.

Proposition 2.

The bisectors of the angles of a triangle are concurrent.

Let ABC be a △; it is required to prove that the bisectors of the angles of this triangle are concurrent.



Bisect ∠s. B,C by BI, CI which meet at I.

Join IA; *then IA shall bisect* $\angle A$

From I, draw IP, IQ, IR perpendiculars to the sides.

Proof. 1. $\because$ BI is the bisector of $\angle B$, $\therefore$ it is the locus of points equidistant from AB and BC. $\therefore IP = IQ$.

2. Similarly $IQ = IR$.

$\therefore IP = IR$ *i.e.*, I is a point equidistant from AB and AC.

3. And hence it must lie on the bisector of $\angle A$, *i.e.*, IA bisects $\angle A$.

Hence the bisectors of the three $\angle$s. of the $\triangle$ meet at I.

NOTE.—I is the **in-centre** of the $\triangle$ ABC.

Exercises.

5. Prove that $AP + BQ + CR = PB + QC + RA = s$, where $2s = a + b + c$.

6. Prove that $AP = AR = s - a$; $CR = CQ = s - c$; $BP = BQ = s - b$.

7. The in-centre of an equilateral triangle coincides with its circum-centre.

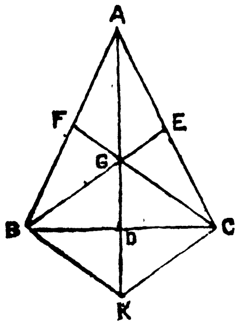
8. Prove that the in-radii of similar triangles are in the ratio of their corresponding sides.

Proposition 3.

The medians of a triangle are concurrent.

Let ABC be a triangle.

Draw the medians BE and CF and let them meet at G. Join AG.



It is required to prove that AG produced bisects BC at D.

Through C, draw CK ∥ BE, meeting AD produced at K.

Join BK.

Proof. **1.** In △ ACK, E is the mid-point of AC and EG ∥ CK (*Cons.*)

∴ G is the mid-point of AK.

2. Again in △ABK, G is the mid-point of AK (*Proved.*)

And F is the mid-point of AB (*Cons.*) ∴ GF, *i.e.*, CF is ∥ BK.

3. Hence GBKC is a parl^m and its diagonals bisect each other; ∴ D is the mid-point of BC; *i. e.*, AD is a median.

∴ The medians of the triangle meet at G.

Note.—The point G is called the **centroid** of the △ ABC.

Exercises.

9. Prove that G is a point of trisection of each median.

10. Given the three medians of a triangle; construct it.

11. Prove that $AG^2+BG^2+CG^2=\frac{1}{3}(a^2+b^2+c^2)$.

12. The base and area of a triangle being given, find the locus of its centroid.

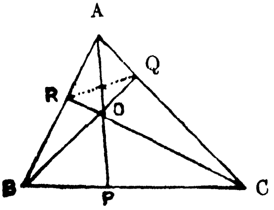
13. The base and vertical angle of a triangle being given, show that the locus of the centroid is an arc of a circle.

ORTHOCENTRE AND PEDAL TRIANGLE.

Proposition 4.

The three altitudes of a triangle are concurrent.

Draw BQ, CR ⊥ to AC, AB; let them meet at O. Join AO and produce it to meet BC in P.



We have to prove that AP *is* ⊥ *to* BC.

Join RQ.

Proof. 1. ∵ ∠*s*. ARO, AQO are rt. ∠*s*., ∴ A,R,O,Q are concyclic;

2. Hence ∠ AQR = ∠ AOR, being in the same segment, = ∠ POC, being vertically opposite.

3. Again, on the same base BC, there are two Δ*s* BRC, BQC having their vertical ∠*s* R, Q = one rt. angle each.

$\therefore$ B, R, Q, C are concyclic.

4. $\therefore \angle RQB = \angle RCB$, being in the same segment, but $\angle RQA = \angle POC$ (step 2).

5. $\therefore \angle RCB + \angle POC = \angle RQB + \angle RQA$

$$= \angle AQB$$

$$= \text{one rt. angle};$$

$\therefore \angle OPC$ is a rt. angle.

Hence AP is perpendicular to BC, and the three altitudes are concurrent.

Def.—The point O, where the three altitudes meet is called the **orthocentre** and the triangle PQR, formed by joining the feet of the perpendiculars is called the **pedal** or **orthocentric** triangle.

Exercises.

14. Every two sides of the pedal triangle are equally inclined to that side of the original triangle in which they meet. Hence shew that the orthocentre of a triangle is the incentre of the pedal triangle.

15. Find the angles of the pedal triangle in terms of those of the original triangle.

16. Any one of the 4 points O, A, B, C is the orthocentre of the triangle formed by joining the other three.

17. The distance of each vertex of a triangle from the orthocentre is double of the perpendicular drawn from the circumcentre to the opposite side.

18. If AO produced meets the circumcircle at G, prove that P is the middle point of OG.

19. The base BC and the opposite angle A being given, show that OA is of constant length.

Proposition 5.

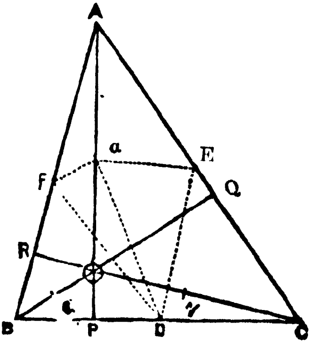
NINE POINTS CIRCLE.

In any triangle (ABC) *the following nine points are concyclic :—(i)* D,E,F *the middle points of the sides ; (ii)* P,Q,R *the feet of the perpendiculars from the vertices to the opposite sides; (iii)* α, β, γ, *the middle points of the joints of the ortho-centre* (O) *to the vertices of the triangle.*

Join αF, αE, αD, DF and DE.

Proof. 1. ∵ α, E are the mid-points of AO and AC,
∴ αE is ∥ OC.

Similarly, ED is ∥ AB.



2. But ∵ ∠ between OC and AB *i.e.*, ∠αR = a rt. angle.

∴ ∠ between αE and ED, *i.e.*, ∠ αED is a rt. ∠.

3. Similarly ∠ αFD is a rt. ∠. And ∠αPD is also a rt. ∠.

4. ∴ if on αD as diameter, a circle is described it must pass through F, P, E ; *i. e.*, the circle passing through D, E, F, must also pass through α and P.

5. Similarly it can be shown that the circle through D, E, F must pass through β, γ and Q, R.

i.e., the nine points α, β, γ; D, E, F; P, Q, R are concyclic.

Def. This circle is called the "**nine points**" circle and its centre is called the **mid-centre** of the triangle.

NOTE.—It has been proved that αD and $\therefore$ βE and γF are all diameters of the nine points circle. *Hence these lines intersect at the mid-centre of the triangle,*

Simple properties of the nine-points circle.

Proposition 6.

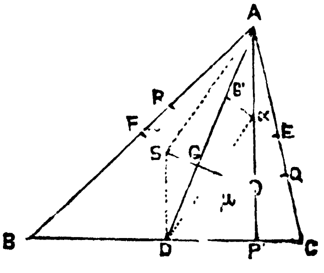
(*i*). *The mid-centre* (μ) *is the mid-point of the join of the Orthocentre* (O) *and the circumcentre* (S).

(*ii*). *The radius* (μD) *of the nine points circle is half the circumradius* (SA).

(*iii*). *The centroid* (G), *the circumcentre* (S), *the mid-centre* (μ), *and the orthocentre* (O) *are all collinear.*

(*i*). The several letters in the figure denote the same points as in the preceding figure. Let S be the circumcentre. Join OS and SD.

Proof 1. $\because$ S is the centre of a circle, of which BC is a chord and $\because$ D is the mid-point of BC, $\therefore$ SD is perp. to BC, $\therefore$ AP and SD are parallel.



2. Again ∵ PD is a chord of the nine-points circle.

∴ its centre must lie on the right bisector of PD.

3. But ∵ this right bisector, OP and SD are all parallel, ∴ the right bisector of PD passes through the middle point of OS.

4. Similarly, the mid-centre must lie on the right bisectors of QE and FR, and these lines, similarly as before, pass through the mid-point of OS.

Hence μ, the middle point of OS is the mid-centre.

(ii) Proof 1. Hence αD has been proved to be a diameter of the nine-points circle (Prop. 5), ∴ it must pass through μ, the mid-point of OS, which has been proved to be the mid-centre (Prop. 6, *i*).

2. Hence αD and OS bisect each other at μ.

∴ the two $\triangle$s $\mu\alpha$O, μSD are evidently congruent,

i.e., SD$=\alpha$O$=\alpha$A;

3. Also SD is ∥ to αA, being both perpendicular to BC

∴ Dα =SA.

4. But Dα is a diameter of the nine-points circle and SA is the radius of the circumcircle.

∴ the radius of the nine-points circle is equal to half the circumradius.

(iii) Proof 1. Join AD; draw αG′ ∥ OS, cutting AD . at G′.

In $\triangle$ AOG, ∴ αG′ is drawn ∥ OG from α, the mid-point of AO, ∴ G′ is the mid-point of AG.

2. Again in $\triangle$ αDG′. ∵ μG is drawn ∥ αG′ from μ, the mid-point of αD, ∴ G is the mid-point of G′D.

3. $\therefore$ $AG = 2GD$; i.e., $AG = \frac{2}{3}AD$;

$\therefore$ G is the centroid of $\triangle$ ABC. (Ex 9).

Hence, *the centroid is collinear with the orthocentre, mid-centre and the circum-centre.*

Exercises.

20. Prove that the triangles ABC, OBC, OCA, OAB have the same nine-point circle.

21. Prove that $EF\beta\gamma$, $DE\alpha\beta$, $FD\gamma\alpha$ are rectangles whose diagonals interesect in μ.

22. Prove that $SG : G\mu : \mu O :: 2 : 1 : 3$.

23. The nine-point circle of an equilateral triangle is its in-circle.

Proposition 7.

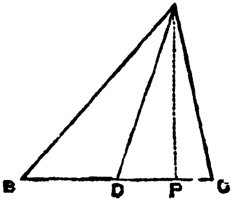
APOLLONIUS' THEOREM.

In a triangle ABC, *if* AD *is a median, then*

$$AB^2 + AC^2 = 2BD^2 + 2AD^2.$$

Draw $AP \perp BC$.

Now whether AP falls within or outside the $\triangle$, of the two $\angle$s ADB, ADC one must be obtuse, and the other acute; let ADB be obtuse.



Proof 1. $\angle$ADB is obtuse and DP is the projection of AD upon BD,

$\therefore AB^2 = BD^2 + AD^2 + 2\,BD \cdot DP$.........(1).

2. Again $\because \angle ADC$ is acute, and DP is the projection of AD upon DC.

$\therefore AC^2 = AD^2 + DC^2 - 2\,CD \cdot DP$.........(2).

3. Adding (1) and (2) and remembering $BD = CD$, we have

$$AB^2 + AC^2 = 2\,AD^2 + 2\,BD^2.$$

Note 1.—Sub-tracting (2) from (1) we get

$$AB^2 - AC^2 = 2\,BC \cdot DP$$

Note 2.—The student will have no difficulty in proving the theorem when AP falls without the triangle or coincides with a side of the triangle.

Exercises.

24 Prove that four times the sum of the squares of the medians of a triangle is equal to three times the sum of the squares of its sides.

25. The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its four sides.

26. If D is the middle point of BC, and A any other point in AB or AB produced, prove that $AB^2 + AC^2 = 2\,AD^2 + 2\,BD^2$.

27. Of two medians of a triangle, that is shorter which bisects the longer side.

23. The locus of the vertex of a triangle which has a given base and the sum of the squares on its sides equal to a given square, is a circle having its centre at the middle point of the base.

29. In a triangle ABC, the base AC is divided at P, so that $m \cdot BP = n \cdot CP$; prove that

$$m \cdot AB^2 + n \cdot AC^2 = m \cdot BP^2 + n \cdot CP^2 + (m+n)\,AP^2.$$

Note.—This is the generalised form of the theorem of Apollonius. It is evident that Prop. 7 is a particular case of this theorem when $m = n = 1$.

30. ABCD is a rectangle and P any point whatever; prove that the squares on PA, PC are together equal to the squares on PB, PD.

31. The squares on the sides of a quadrilateral are together greater than the squares on the diagonals by four times the square on the straight line joining the mid-points of the diagonals.

Proposition 8.

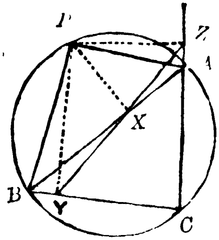
THE SIMSON LINE.

The feet (X, Y, Z) of the perpendiculars, from any point (P) *on the cir-cumcircle of a triangle* (ABC), *to the sides of the triangle, lie on a straight line.*

To prove that Y, X, Z, are collinear. Join YX, XZ, PA, PB.

Proof 1. ∵ APBC is a cyclic quadrilateral,

∴ the exterior. ∠ PBY = opp. internal ∠ PAC.. ......*(i)*.



2. Again ∵ ∠ PYB + ∠ PXB = two rt. ∠*s* (Hyp).
∴ P, Y, B, X are concyclic;

3. ∴ ∠ PBY = ∠ PXY (in the same segment)........*(ii)*.

Hence ∠ PAC = ∠ PXY from *(i)* and *(ii)*.

4. Again, on the same base AP, there are two △s AZP, AXP having their vertical angle AZP, AXP having their vertical angles AZP, AXP equal (being rt. ∠*s*).

∴ A, Z, X, P are concyclic,

and ∴ ∠PAZ + ∠PXZ = two rt. ∠s.

5. But ∠PAZ = ∠PXY (step 3)

∴ ∠PXY + ∠PXZ = 2 rt. ∠s, hence XYZ is a straight line.

Note.—The line YXZ is called the **pedal line** or **Simson line** of the triangle ABC for the point P.

Exercises.

32. (Converse of Prop. 8.) If from any point P. perpendiculars PX, PY, PZ be drawn to the sides of a triangle, and if X, Y, Z, are collinear, the point P must lie on the circumcircle of the triangle.

33. AD is prependicular to BC, a side of a triangle ABC; AD meets the circumcircle in Q; prove that the Simson line of Q is parallel to the tangent at A.

34. The pedal line of any point bisects the line joining the point to the orthocentre.

35. I is the centre of the triangle ABC; AI meets the circumcircle in R; prove that the simson line of R bisects BC.

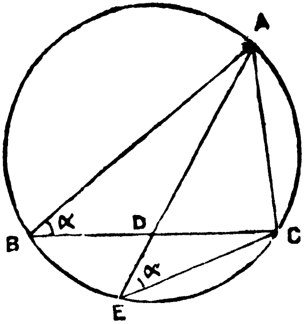
36. P is a point on the circumcircle of the triangle ABC The pedal line of P cuts AC and BC in X and Y. L is the of the prependicular from P on the pedal line. Prove that the rectangles PL. PC and PX. PY are equal.

37. What is the simson line of A in the figure of Prop. 8?

Proposition 9.

If ABC be a triangle and AD, the bisector of the angle A *then* AB . AC = BD . CD + AD².

Produce the bisector AD to meet the circumcircle of △ ABC in E. Join EC.



Proof. 1. In △s ABD, EAC, ∴ ∠ BAD = ∠ EAC (Hyp.) ∠ α = ∠ α', being in the same segment;

2. ∴ the △s are similar, and ∴ $\frac{BA}{AD} = \frac{EA}{AC}$.

3. ∴ AB . AC = AD . AE

$$= AD\,(AD + DE)$$
$$= AD^2 + AD\,.\,DE$$
$$= AD^2 + BD\,.\,DC.$$

∵ BC and AE are two intersecting chords of the circumcircle.

Exercises.

38. If the external bisector of the angle A meets BC produced at D′, then AB . AC = BD′ . CD′ − AD′².

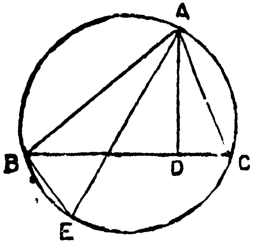
39. Prove that $AD^2 = \frac{4\,bc}{(b + c)^2}\,s\,(s - a)$, where a, b, c are the sides of the triangle and s is the semiperimeter.

40. Prove that ED. EA = EC².

Proposition 10.

If ABC *is a triangle, and* AD *the perpendicular from* A *to* BC, *and* AE, *a diameter of the circle, then*

AB . AC = AD . AE.



Join EB.

Proof. **1.** In the △*s* ABE, ACD ∴ ∠AEB = ∠C (in the same segment);

rt. ∠D = rt. ∠ABE (in the semi-circle).

2. ∴ the △*s* are similar;

Hence $\frac{AC}{AD} = \frac{AE}{AB}$; ∴ AB . AC = AD . AE.

Exercises.

41. From this proposition prove that the circumradius = $\frac{a\,b\,c}{4\Delta}$ where a, b, c denote the sides of the triangle and Δ, the area of the triangle.

42. From the formula in Ex. 41, find the value of R when the sides of the triangle are 21 ft., 20 ft., 13 ft. Draw the triangle on a convenient scale and check your result by measurement.

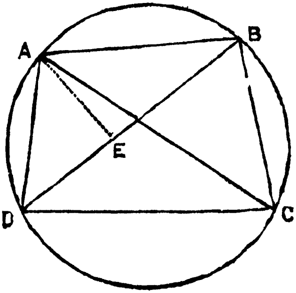
Proposition 11.

PTOLEMY'S THEOREM.

The rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles contained by its opposite sides.

It is required to prove that AC . BD = AB . CD + AD. BC.

Make ∠DAE = ∠BAC.



Proof. **1.** Now in the two △*s* DAE, BAC,
∵ ∠DAE = ∠BAC (Cons.); ∠ADE = ∠ACB (in the same segment);

2. ∴ the △*s* are similar. ∴ $\frac{AD}{DE} = \frac{AC}{BC}$;

or AD . BC = AC . DE.........(*i*).

3. Again in the two △*s*. DAC, BAE,
∴ ∠DAC = ∠BAE (∠*s* DAE and BAC being equal);
and ∠ACD = ∠ABE (in the same segment);

4. ∴ the △*s* are similar.

$$\therefore \frac{CD}{CA} = \frac{BE}{BA};$$

$$\therefore CD \,.\, AB = AC \,.\, BE \ldots\ldots\ldots (ii).$$

5. Adding (i) and (ii) we have,
$AD \,.\, BC + CD \,.\, AB = AC\,(DE + BE) = AC \,.\, BD.$

Exercises.

43. The rectangle contained by the diagonals of a quadrilateral is less than the sum of the rectangles contained by its opposite sides unless the quadrilateral is cyclic.

44. P is the point on the arc BC of the circumcircle of an equilateral triangle ABC. Shew that PB+PC=PA.

45. What does Ptolemy's theorem become in the particular case when two vertices of the quadrilateral coincide?

46. If the diagonals of a cyclic quadrilateral be at right angles, the sum of the rectangles contained by its opposite sides is equal to twice the area of the quadrilateral.

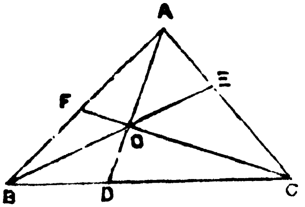
47. If C is the middle point of an arc of a circle whose chord is AB, and D is any point on the conjugate arc; shew that AD+DB : DC : : AB : AC.

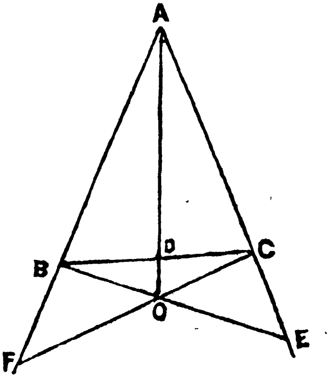
Proposition 12.

CEVAS' THEOREM.

If through the vertices of a triangle ABC *three concurrent lines be drawn to meet the opposite sides* BC, CA, AB *in* D, E, *F respectively, to prove that*

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$$





Let O be the point of concurrence

Proof. 1. Now (*i*) $\frac{BD}{DC}=\frac{\triangle\ ABD}{\triangle\ ACD}$

also (*ii*) $\frac{BD}{DC}=\frac{\triangle\ BOD}{\triangle\ DOC}$.

2. $\therefore$ $\frac{BD}{DC}=\frac{\triangle\ ABD}{\triangle\ ACD}=\frac{\triangle BOD}{\triangle DOC}$

$$=\frac{\triangle\ ABD\pm\triangle\ BOD}{\triangle\ ACD\pm\triangle\ DOC}$$

$$=\frac{\triangle\ AOB}{\triangle\ AOC}\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

3. Similarly $\frac{CE}{EA}=\frac{\triangle\ BOC}{\triangle\ AOB}\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$

and $\frac{AF}{FB}=\frac{\triangle\ AOC}{\triangle\ BOC}\ \ldots\ldots\ldots\ldots\ldots\ldots(3)$

Hence multiplying (1), (2) and (3)

we have

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1\ldots\ldots\ldots\ldots\ldots\ldots(A)$$

Note. The result (A) may be written as follows:—

$$BD\,.\,CE\,.\,AF=DC\,.\,EA\,.\,FB.$$

Hence this theorem may be enunciated thus:—

If three concurrent straight lines are drawn from the angular points of a triangle to meet the opposite sides, then the product of three alternate segments taken in order is equal to the product of the other three segments.

Proposition 13.

CONVERSE OF CEVA'S THEOREM.

If D, E, F *be points in the sides* BC, CA, AB *of the triangle* ABC *such that*

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$$

then AD, BE, CF *are concurrent.*

Proof. 1. Suppose BE, CF meet in O, and suppose AO (produced if necessary) cut BC in a point D′.

2. Then since AD′, BE, CF are concurrent ∴ by the previous theorem

$$\frac{BD'}{D'C}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$$

3. But $$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1\ldots\ldots(Hyp.)$$

4. ∴ $$\frac{BD'}{D'C}=\frac{BD}{DC}$$

5. Hence D and D′ coincide (since the point of internal division of a st. line in a given ratio is unique)

$\therefore$ AD, BE, CF are concurrent.

Note :—The converse theorem may also be stated thus :—

If three straight lines drawn from the vertices of a triangle cut the opposite sides so that the product of three alternate segments taken in order is equal to the product of the other three segments, then the three straight lines are concurrent.

Exercises.

48. The three medians of a triangle are concurrent.

49. The three altitudes of a triangle are concurrent.

50. The bisectors of the three angles of a triangle are concurrent.

51. The straight lines joining the vertices of a triangle to the points of contact of the inscribed circle with the opposite sides are concurrent.

52. The escribed circles of a triangle touch the sides opposite to the angles A, B, C in A', B', C' respectively; prove that AA', BB', CC' are concurrent.

53. The lines from the vertices of a triangle to the points of contact of an escribed circle are concurrent.

54. If AF : FB = AE : EC, show that the line joining A to the intersection of BE and CF is a median.

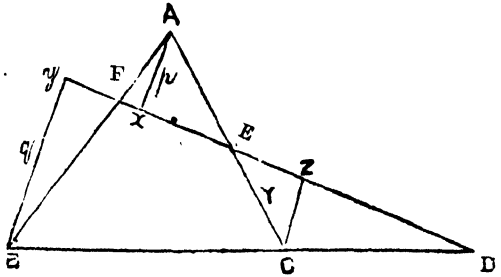
55. x, x' are points on BC such that $Bx = x'c$. The points y, y' ; z, z' are similarly related pairs of points on CA, AB. If Ax, By, Cz are concurrent, so also are Ax', By', Cz'.

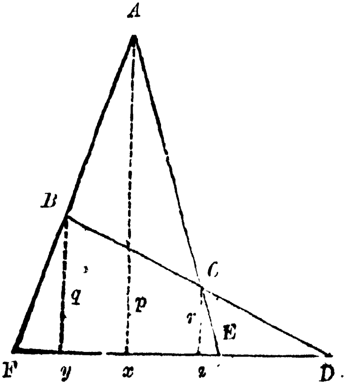
Proposition 14.

MENELAUS' THEOREM.

If a transversal cut the sides BC, CA, AB *of a triangle* ABC *in* D, E, F *respectively, to prove that*

$$\frac{DB}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1\text{.........}(\textit{in magnitude})$$





Proof. 1. Suppose the lengths of the perpendiculars from A, B, C upon the transversal to be p, q, r respectively.

2. $\because$ $\triangle$s AFx, BFy are similar

$$\therefore \frac{AF}{FB} = \frac{p}{q} \text{ ....(1)}$$

3. Similarly, by similar triangles,

$$\frac{CE}{EA} = \frac{r}{p} \text{ ....(2)}$$

$$\text{and } \frac{DB}{DC} = \frac{q}{r} \text{ ....(3)}$$

4. Hence multiplying (1), (2) and (3) we have

$$\frac{DB}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{q}{r} \times \frac{r}{p} \times \frac{p}{q} = 1$$

Note :—From the above we have

$$DB \,.\, CE \,.\, AF = DC \,.\, EA \,.\, FB.$$

which may be expressed as follows :—

If a straight line in drawn to cut the sides or the sides produced of a triangle the product of three alternate segments taken in order is equal to the product of the other three segments.

Proposition 15.

CONVERSE OF MENELAUS' THEOREM.

If D, E, F *be points in the sides* BC, CA, AB *respectively of the triangle* ABC *such that*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

then D, E, F *are collinear.*

Proof. **1.** Join EF and suppose it produced to meet BC in D′.

2. Now $\because$ D′, E, F are collinear

$$\therefore \frac{BD'}{D'C} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1 \text{ ....(1)}$$

3. But $$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1 \text{ ....(2)}$$

4. Whence

$$\frac{BD'}{D'C}=\frac{BD}{DC}$$

5. ∴ D and D′ coincide (since the point of internal or external division of a straight line in a given ratio is unique.)

6. Hence D, E, F are collinear.

Note:—The converse theorem may also be put in the following form:—

If three points are taken in two sides of a triangle and the third side produced or in all the three sides produced so that the product of three alternate segments taken in order is equal to the product of the other three segments, the three points are collinear.

Remark—In proving Ceva's and Menelaus' theorems and their converses, we have taken no account of the signs of the lines. If the signs be taken into account, some of the ratios would be negative, and in Menelaus' Theorem the product of the ratios would be negative. The student is referred to the note on "**Different signs of lines**" preceding Euler's Theorem.

Exercises.

56. The bisectors of the three external angles of a triangle meet the opposite sides in three points, which are collinear.

57. The bisectors of angles B and C meet the opposite sides in Q, R, and QR meets BC in P; prove that AP is the exterior bisector of the angle A.

58. α, β, γ are the mid-points of the sides of a triangle ABC; Aα meets $\beta\gamma$ in P; CP meets AB in Q. Show that AQ$=\frac{1}{3}$ AB.

59. If the tangents at A. B, C of the circumcircle of the triangle ABC meet the opposite sides in D, E, F respectively, shew that $BD : CD = BA^2 : AC^2$. Hence prove that D, E, F are collinear.

Definition.—If every point on a line or group of lines satisfies a given geometrical condition and no other point does so, then that line or group of lines is called the **locus** of the point satisfying that condition.

In order to completely establish that a certain line or group of lines is the locus of points satisfying a given condition we must prove **two** things:—

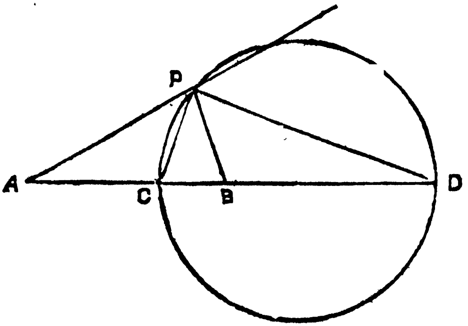
(*i*). *Any point that satisfies the given condition lies on the line ;*

(*ii*). *Any point on the line satisfies the given condition.*

Proposition 6.

APOLLONIAN LOCUS.

The locus of a point (P), *the ratio of whose distances* (PA, PB) *from two given points* (A, B) *is equal to a given ratio* $\left(\frac{l}{m}\right)$, *is a circle.*



Let $\frac{PA}{PB} = \frac{l}{m}$;

it is required to prove that P *lies on a circle.*

Let the internal and external bisector of ∠ APB meet AB in C and D.

Proof. 1. Then ∵ PC bisects ∠ APB;

$\therefore \frac{AC}{BC} = \frac{AP}{PB} = \frac{l}{m}$, given ratio.........(Hyp).

2. Similarly $\frac{AD}{BD}$ = the given ratio.

3. But ∵ A and B are fixed points, ∴ C and D are also fixed.

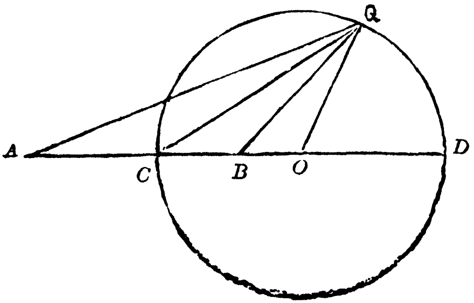
4. Also ∵ PC and PD bisect the ∠ APB internally and externally;

∴ CPD is a rt. ∠.

Hence, P lies on a circle described on CD as diameter, since the ∠ in a semicircle is a rt. ∠.

(*ii*). If Q be any other point taken on this circle, to prove

$$\frac{AQ}{QB} = \frac{l}{m}$$



Proof. 1. Now (*i*) $\frac{AD}{DB}=\frac{l}{m}$

(*ii*) $\frac{AC}{CB}=\frac{l}{m}$

2. $\therefore \frac{AD}{DB}=\frac{AC}{CB}=\frac{AD-AC}{DB-CB}=\frac{CD}{2BO}=\frac{2DO}{2BO}=\frac{DO}{BO}=\frac{QO}{BO}$

3. Again $\frac{AD}{DB}=\frac{AC}{CB}=\frac{AD+AC}{DB+CB}=\frac{2AO}{2CO}=\frac{AO}{CO}$

4. $\therefore \frac{AO}{CO}=\frac{QO}{BO}$ (from steps 2 and 3)

$\therefore \frac{AO}{QO}=\frac{CO}{BO}=\frac{DO}{BO}$.

5. Now in the two triangles, AOQ, BOQ

$$\frac{AO}{OQ}=\frac{OQ}{BO}$$

and the included $\angle$ BOQ is common.

6. $\therefore$ the two triangles are similar

Hence $\angle BQO=\angle A$..................................(1)

7. Again $\angle OQC=\angle OCQ$

$=\angle A+\angle CQA$................(2)

8. $\therefore \angle OQC-\angle BQO=(\angle A+\angle CQA)-\angle A.$

i. e., $\angle BQC=\angle CQA$

$\therefore \angle BQA$ is bisected by CQ

$\therefore \frac{AQ}{QB}=\frac{AC}{CB}=\frac{l}{m}$

Hence the circle described on CD as diameter, is the locus of a point P which moves in such a manner that the ratio $\frac{AP}{PB}$ = constant.

Exercises.

60. Find a point whose distances from three given points shall be to one another in the ratio of three given lengths.

61. Construct a triangle having given one side, the angle opposite to that side and the ratio of the other two sides.

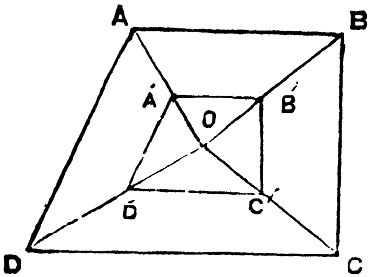
62. A and B are fixed points. Another point P moves so that PA is always equal to 3 PB. Prove that P lies on a circle whose diameter is equal to $\frac{3}{4}$ AB.

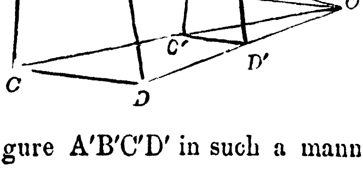
Proposition 17.

HOMOTHETIC FIGURES.

Any two similar rectilineal figures may be so placed that the lines joining their corresponding vertices are concurrent.

Let ABCD, A′B′C′D′ be similar figures.





Place the figure A′B′C′D′ in such a manner that A′B′ is

$\parallel$ AB; then $\because \angle A = \angle A'$ and $\angle B = \angle B'$, $\therefore$ A'D' and B'C' shall be $\parallel$ AD and BC respectively.

Similarly it follows that D'C' is $\parallel$ DC.

It is now required to prove that when corresponding sides of the given figures are $\parallel$ *then* AA', BB', CC', DD' *are concurrent.*

Join AA', and divide it *externally* at O in the ratio of AB : A'B'; *i. e.*, AO : A'O = AB : A'B'.

Join OB, OB'; *it will be proved that* OB *and* OB' *are in the same straight line.*

Proof 1. In the $\triangle$*s* OAB, OA'B', $\because$ AB is $\parallel$ A'B', $\therefore \angle OAB = \angle OA'B'$; and by construction

OA : OA' = AB : A'B';

2. $\therefore$ the $\triangle$*s* OAB, OA'B' are equiangular,

i. e., $\angle AOB = \angle A'OB'$, and hence OB, OB' are in the same straight line; *i. e.*, BB' passes through O.

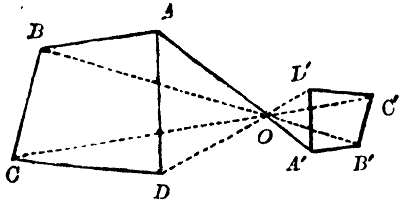
3. Similarly CC', DD' may be shown to pass through O, $\therefore$ AA', BB', CC', DD' are concurrent.

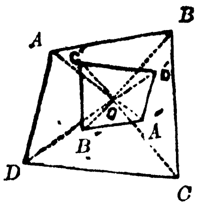
Note 1.—The two figures might be placed outside each other, but observe that in both the cases, the joining lines AA', BB', CC', DD' are all divided externally at O in the ratio of any pair of corresponding sides of the given figures.

Note 2.—In placing the given figures so that the corresponding sides are parallel two cases may arise:

(*i*) the corresponding sides are parallel and drawn in the same direction, as in the figures given above;

(*ii*) the corresponding sides are parallel but drawn in opposite directions as in Figures given below .—





In this case, O divides AA′, BB′, &c., *internally* in the ratio of corresponding sides.

Def.—The point O is called a *centre of similarity* or *homothetic centre ;* and similar figures thus placed are said to be *homothetic.*

Exercises.

63. Inscribe an equilateral triangle in a given triangle, having one side parallel to a given side of the triangle.

64. In a given regular pentagon inscribe a square, so that one side of the square may be parallel to a side of the pentagon.

65. In a given triangle inscribe a square.

Note. The primary idea of similarity is *likeness of shape.*

In order that two polygons may be similar, the two essential conditions are :—

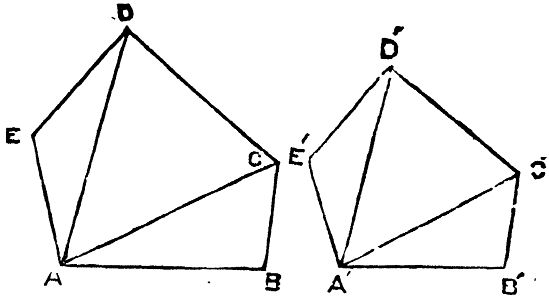
(*i*). For every angle in one of the polygons, there must be an equal angle in the other.

(*ii*). The ratio of corresponding sides (which lie between pairs of equal angles) are also equal.

In the case of two triangles any one of the conditions stated above involves the other but such is not the case with other polygons.

Proposition 18.

Similar polygons can be divided into the same number of similar triangles; and the lines joining corresponding ertices are proportional.



Let ABCDE, A′B′C′D′E′ be two similar figures, the vertex A corresponding to A′, B to B′ and so on.

Join AC, AD; A′C′, A′D′.

It is required to prove that,

(*i*). △*s* ABC, A′B′C′ are similar; so also other pair;

(*ii*). AB : A′B′ = AC : A′C′ = AD : A′D′.

Proof. 1. ∵ The figures ABCDE, A′B′C′D′E′ are similar ∴ (*a*). ∠ B = ∠ b′

and (*b*). AB : BC = A′B′ : B′C′.

2. Hence the △*s*. ABC, A′B′C′ are similar.

3. Again ∵ (*a*). ∠ ACB = ∠ A′C′B′
(△*s* ABC, A′B′C′, being similar

and (*b*). whole ∠ BCD = whole ∠ B′C′D′
(Figs ABCDE, A′B′C′D′E′ being similar.

$\therefore$ the remaining $\angle$ ACD = remaining $\angle$ A′C′D′.

4. Now $\because \dfrac{AC}{CB} = \dfrac{A'C'}{C'B'}$

and $\dfrac{BC}{CD} = \dfrac{B'C'}{C'D'}$.

5. $\therefore$ multiplying $\dfrac{AC}{CD} = \dfrac{A'C'}{C'D'}$

and $\angle$ ACD = $\angle$ A′C′D′... (Step 3).

Hence $\triangle$s. ACD, A′C′D′ are similar.

6. Similarly it can be shown that

$\triangle$s ADE, A′D′E′ are similar.

(ii). **Proof.** 1. $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$

($\triangle$s. ABC, A′B′C′ being similar.

2. $\dfrac{AC}{A'C'} = \dfrac{AD}{A'D'}$ ($\triangle$s. ACD, A′C′D′, being similar.)

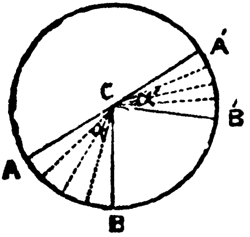
8. $\therefore$ $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'} = \dfrac{AD}{A'D'}$.

Note.—The polygons may also be divided into the same number of similar triangles by joining their vertices to a point within the polygons. We leave the construction and proof as an exercise to the student.

Proposition 19.

Arcs of a circle are proportional to the angles subtended by them at the centre.

Let the arcs AB, A′B′ subtend angles α, α' at the centre O of the circle.



To prove that arc AB : *arc* A′B′ $= \angle\alpha : \angle\alpha'$.

Suppose the arc AB : arc A′B′ $= m : n$;......(1)

so that if the arc BA is divided into m equal parts, then the arc A′B′ may be divided into n such equal parts. Draw the radii to the points of division of the arcs AB, A′B′.

Proof. 1. Now $\angle$s α, α' are divided into $\angle$s, which stand on equal arcs and are, therefore, all equal.

2. And of these equal $\angle$s, α contains m; α' contains n; $\therefore \angle\alpha : \angle\alpha' = m : n$.

3. $\therefore$ arc AB : arc A′B′ $= \angle\alpha : \angle\alpha'$.......from (1).

Cor. 1. Since angle at the centre is double of the corresponding angle at the circumference, therefore arcs, of a circle are also proportional to the angles subtended by them at the circumference.

Cor. 2. Since in a circle, sectors which have equal angles are equal, it may be proved as above that the sector ACB : the sector A′CB′ = arc AB : arc A′B′.

Cor. 3. In *equal* circles, angles (whether at the centres or circumferences) and sectors have the same ratio as the arcs on which they stand.

Definition. Two quantities are said to be *incommensurable* when they have no common measure. The side and diagonal of a square are such quantities. An *incommensurable number* is one whose value can not be expressed as the ratio of two integers. The value of such a number can not be found exactly, it can be found only approximately. $\sqrt{2}$, π, e are examples of such numbers.

The above proof is applicable to the case when arcs AB and A′B′ are *commensurable* when the arcs are incommensurable, the proof is more difficult as involving the theory of limit. In elementary geometry this difficulty is got rid of by saying that by taking the unit of measure infinitely small we can make an incommensurable magnitude *practically* commensurable.

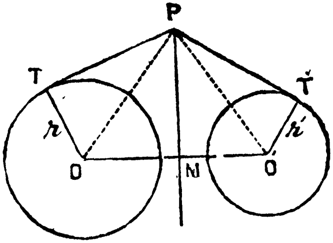
THE RADICAL AXIS.

Def.—The locus of points from which the tangents drawn to two given circles are equal is called the **Radical Axis** of the two circles.

Proposition 20.

To find the radical axis of two given circles.

Let O and O′ be the centres of the given circles, whose radii are r and r': and let P be any point such that the tangent PT drawn to the circle (O) is equal to the tangent PT′ drawn to the circle (O′).



It is required to find the locus of P.

Join OO′, PO, PO′, OT, O′T′.

From P draw PM perp. to the line of centres OO′.

Proof. **1.** Now ∵ PT = PT′, ∴ $PT^2 = PT'^2$.

2. But $PT^2 = OP^2 - r^2$, since ∠ T is a rt. ∠.

Similarly $PT'^2 = O'P^2 - r'^2$.

3. ∴ $OP^2 - r^2 = O'P^2 - r'^2$;

∴ $PM^2 + OM^2 - r^2 = PM^2 + O'M^2 - r'^2$ [Pythagoras' theorem.]

4. Hence $OM^2 - r^2 = O'M^2 - r'^2$.

i. e., $OM^2 - O'M^2 = r^2 - r'^2$;

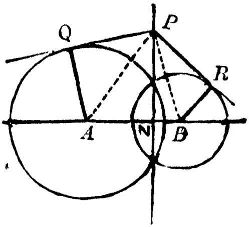
5. ∴ OO′ is a fixed st. line and r, r' fixed lengths, ∴ M is a fixed point.

Hence all points from which equal tangents can be drawn to the two circles lie on the straight line which cuts OO′ at right angles, such that the difference of the squares on the segments of OO′ is equal to the difference of the squares on the radii.

Hence PM is the required radical axis.

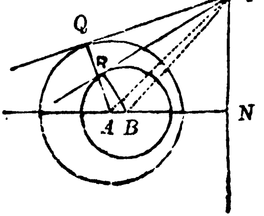
Cor. 1. By taking any point in PM and by retracing the steps of the proof, it may be shown that tangents drawn from that point to the two circles are equal.

Cor. 2. When the two circles intersect, it is evident that tangents drawn to two intersecting circles from any point in the common chord produced are equal. The common chord, according to the definition is *not* the radical axis. The part of the common chord produced either side is the radical axis.



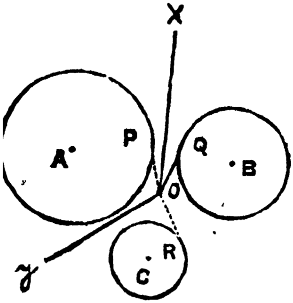
Cor. 3. If K is the middle point of OO′, it may be shewn that $OM^2 - O'M^2 = 2\ OO' \cdot KM$, thus giving the position of the point M with respect to the middle point, of the line of centres.

Note.—The following figure shows the position of the radical axis in the case when one of the circles falls within the other:—



Proposition 21.

The radical axes of three circles taken in pairs are concurrent.



Let (A), (B), (C) be the three circles. Suppose OX is the radical axis of circles (A) and (B), and Oy the radical axis of circles (A) and (C).

It is required to prove that the radical axis of circles (B) *and* (C) *passes through* O.

From O draw OP, OQ, OR tangents to three circles.

Proof. 1. Then ∵ O is a point on the radical axis of circles (A) and (B),

∴ OP=OQ.

2. Again ∵ O is a point on the radical axis of circles (A) and (C),

∴ OP=OR.

∴ OQ=OR ; *i. e.*, O is a point on the radical axis of circles (B) and (C).

3. ∴ the radical axis of (B) and (C) passes through O, and hence the three radical axes are concurrent.

Note.—It will be found that the point O is either without or within all the circles. If the given circles intersect in such a way that O is within them all, then the radical axes are the common chords of the circles taken in pairs, and it can be easily shewn that these common chords are concurrent.

Def. The point of intersection of the radical axes of three circles taken in pairs is called the **radical centre.**

Exercises.

66. Prove the validity of the following construction for the radical axis of two circles. Draw any circle to cut one of the circles in P, P′ and the other in Q, Q′, produce PP′, QQ′ to meet at O; draw ON perp. to the line of centres. Then ON is the radical axis of the two circles.

67. Shew that the radical axis of two circles bisects any of their common tangents.

68. If three circles touch one another, taken two and two shew that their common tangents at the points of contact are concurrent.

69. If circles are described on the three sides of a triangle as diameters, their radical centre is the orthocentre of the triangle.

70 The difference of the squares on the tangents drawn from any point to two circles is equal to twice the rectangle contained by the line joining their centres and the perpendicular from the given point on their radical axis.

71. In the case of each of the following pairs of circles, find the ratio in which their radical axis divides the lines of centres. Make sketches of figures R and R′ are the radii of the circles and d the distance between their centres:—

(*i*) $R = 4''$, $R' = 2''$, $d = 7''$;
(*ii*) $R = 3''$, $R' = 2''$, $d = 5''$;
(*iii*) $R = 2''$, $R' = 1''$, $d = 2{\cdot}5''$;
(*iv*) $R = 2''$, $R' = 1''$, $d = 1''$,
(*v*) $R = 1{\cdot}8''$, $R' = 1{\cdot}4''$, $d = 0$.

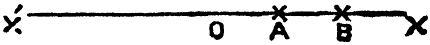
72. What is the radical axis of two circles which touch each other?

73 If the centres of three circles are collinear, where is their radical centre? Where is the radical centre of three circles, two of which are concentric?

Different Signs of "Lines."

The algebraical signs of + and − may be very conveniently introduced into geometrical investigations and in order to do this we must assume that a geometrical line possesses both magnitude and sign.

Let us take an indefinite straight line passing through



the point O, extending on both sides of it—the point O is called the origin.

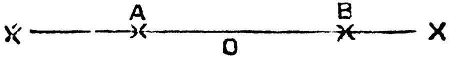
We assume, for the sake of convenience that lengths measured in the direction OX are positive and hence lengths measured in the opposite direction OX′ are negative.

Again, if the line AB measured from A to B is reckoned positive, then the line BA, measured from B to A is to be considered negative. Thus $AB = -BA$.

It is clear from the above figure that

$$AB = OB - OA.$$

The above relation holds good even when the point A is on the opposite side of O, for in that case,



$$OB - OA = OB + AO$$
$$= AO + OB = AB.$$

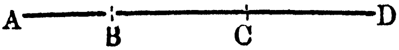
We shall apply this convention in proving the following theorem.

Proposition 22.

EULER'S THEOREM.

If A, B, C, D be any four collinear points, then

$$AB \,.\, CD + AD \,.\, BC + AC \,.\, DB = 0.$$



Take A as origin, and let $AB = x$, $AC = y$, $AD = z$.

Then $CD = AD - AC = z - y$;

$BC = AC - AB = y - x$;

$DB = -BD = -(AD - AB) = -(z - x)$.

$\therefore AB \,.\, CD + AD \,.\, BC + AC \,.\, DB$

$$= x(z - y) + z(y - x) - y(z - x)$$
$$= xz - xy + yz - xz - yz + xy = 0.$$

Exercises.

74. If M is the middle point of the line AB and O any other point on the line, then $OM = \frac{1}{2}(OM + OB)$.

75. If A, B, C be any three collinear points, then

$$BC + CA + AB = 0.$$

76. If A,B,C,D be four collinear points, then in whatever way they may be arranged

$$AB + BC + CD + DA = 0.$$

Miscellaneous Exercises.

77. The straight line joining the orthocentre of a triangle to any point on the circumference of its circumcircle is bisected by the pedal of that point.

78. Bisect a quadrilateral by a straight line drawn through an angular point.

79. Every straight line which passes through the extremities of two parallel radii of two fixed circles passes through one or other of two fixed points.

80. Find a point in a given straight line such that the sum of its distances from two given points on the same side of it, shall be a minimum.

81. The pedal triangle is a triangle of maximum perimeter which can be inscribed in the original triangle.

82. Given the base and vertical angle of a triangle, find the locus of the incentre.

88. Prove that the nine points circle touches the inscribed circle.

84. Through one of the points of intersection of two given circles draw the maximum straight line terminated by the circumferences.

85. The base and area of a triangle being given, find the locus of its centroid.

86. The base and vertical angle of a triangle being given show that the locus of the centroid is an arc of a circle.

87. Describe a polygon similar to a given polygon, but of half its area.

88. Construct a triangle, having given its pedal triangle.

89. Prove that the radius of the circle which passes through the middle points of the sides of a triangle is equal to half the radius of the circle which passes through the angular points.

90. ABCD is any quadrilateral, and O the point of intersection of the straight lines joining the middle points of pairs of opposite sides. If P is any point, then

$$PA^2 + PB^2 + PC^2 + PD^2 = OA^2 + OB^2 + OC^2 + OD^2 + 4\,OP^2.$$

91. Find a point such that the feet of the four perpendiculars from it to the sides of a given quadrilateral may be collinear.

92. The four circumcircles of the four triangles formed by four straight lines, no two of which are parallel, have a common point of intersection.

93. Circles are described on the sides of a triangle as diameters, prove that the radical centre of these circles is the orthocentre of the triangle.

94. ABCD is a quadrilateral, and X the middle point of the straight line joining the bisection of the diagonals; with K as centre *any* circle is described, and P is any point on this circle; shew that $PA^2 + PB^2 + PC^2 + PD^2$ is constant, being equal to $XA^2 + XB^2 + XC^2 + XD^2 + 4\,XP^2$.

Hints for the Solution

OF

Miscellaneous Exercises.

77. ABC is a Δ. H is orthocentre. P, any point in arc BC. PL is perpendicular to BC and PM perpendicular to AC produced. ML produced meets HP at Q. AD is perpendicular to BC and meets the circle at E. Join EP, produce it to meet BC produced at K and LM produced at R Join HK, PC. Δ HDK = Δ EDK. angle HKD = angle EKD = Comp. of angle PLM = angle MLC, ∴ RQ ∥ KH. In Δ KLP, RL = RK = RP. Hence &c., &c.

78. ABCD is quadrilateral. Join AC, BD Bisect BD at E. Draw EP ∥ AC meeting BC at P. Join AP. Then AP bisects the given figure. Join AE, CE.

79. The two fixed points are where the line of centres is divided internally and externally in the ratio of the radii.

80. P, Q are the given points. From P draw PA perpendicular to the given line and from PA produced cut off AB = PA. Join BQ cutting the given line at O. Then O shall be the point required.

81. Apply Exs. 14 and 80.

82. Angle A is given, I is incentre. Prove that angle BIC = one rt. angle + $\frac{1}{2}$A and ∴ locus of I is the arc of a segment on BC.

83. The distance between the centres of the circles is equal to the difference of their radii.

84. It is ∥ to line of centres.

85 The altitude is given ; ∴ the locus is a straight line ∥ to the base at a distance of $\frac{1}{3}$ of the altitude from the base.

86. BC the given base. Cut off DB′ = $\frac{1}{3}$ DB and DC′ = $\frac{1}{3}$ DC. Join B′G, C′G. Δ B′GC′ is equiangular with ABC. ∴ locus of G is a segment on B′C′ similar to the segment on BC.

87. ABCD is polygon. Bisect AD in E, draw EL perpendicular to AD and = AE Cut off AD′ from AD and = AL. Join AC. Draw D′C′, C′B′ || DC, CB. AB′CD′ is the required polygon.

88. The incentre of the pedal Δ is the orthocentre of the original Δ.

89. Follows from Ex. 3.

90. Apply Prop. 7 to the Δs PAB, PAC, OAB, OCD, and remember that O is the middle point of the joins of the midpoints of opposite sides.

91. Produce two sides of the given quadrilateral to make a triangle with a third side. Obtain two such triangles. Draw their circumcircles. The point of intersection of these circles is the required point. Apply Simson's line.

92. Draw two of the circles. Take one of the points of intersection; from these draw four perpendiculars upon the four straight lines. Then apply converse of "The simson line."

93. Besides the vertices A, B, C these circles intersect in L, M, N, the feet of altitudes. Hence AL, BM, CN, are the radical axes.

94. Apply Prop. 7 to Δs XAC, XBD; APC, BPD. Add and compare results.